Nakagami-*m* phase-envelope joint distribution

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The *exact* Nakagami-*m* phase-envelope joint distribution is obtained. Apart from the case $m=1$, the phase distribution is not uniform. It bears similar shapes to those of Hoyt and Rice, and coincides with those at the limiting values of their parameters, for which their envelopes also coincide.

Introduction: When the Nakagami-*m* envelope distribution was proposed [1], no information on the phase distribution of the corresponding signal was provided. In fact, to the best of the authors' knowledge, the Nakagami-*m* phase-envelope joint probability density function (pdf) has not been established yet, either by physical argument or by empirical evidence [2–4]. Accordingly, the usual uniform pdf assumption for the phase has been adopted, independently of the parameter m. We conjecture on the inadequacy of the uniform assumption as follows. It is widely known that the Nakagami-*m* distribution approximates both Hoyt, for $m<1$, and Rice, for $m>1$. In fact, for the Hoyt parameter $b=0$, the Rice parameter $k=0$, and the Nakagami parameter $m=1$, these three distributions specialise into the Rayleigh one for which the uniform phase applies. Apart from this very special case, the phase distribution for Hoyt and Rice are by no means uniform. In addition, for $b=\pm1$ and $m=0.5$, Hoyt and Nakagami-*m* envelope distributions coincide with the one-sided Gaussian for which the Hoyt phase distribution are impulses at 0 and π, or at $\pi/2$ and $3\pi/2$. Moreover, for $k\to\infty$ and $m\to\infty$, Rice and Nakagami-*m* envelope distributions also tend to coincide with each other, but the Rice phase in this case tends to impulses at the phase where its dominant component lies. For instance, if the dominant component energy is equally distributed between the in-phase and quadrature components, the corresponding phase impulses appear at $\pi/4$, or $3\pi/4$, or $5\pi/4$, or $7\pi/4$. For consistency, any phase model proposal for the Nakagami-*m* signal must comply with these features. More specifically, it must coincide with those of Hoyt and Rice for the cases in which their envelope distributions also coincide. It is also desirable that for the other cases the phase distributions bear similar shapes as is the case of their envelope distributions. The fulfilment of these conditions may probably be carried out in several ways, but one useful requisite is to maintain the formulation as simple as possible by, for instance, not including additional parameters. In this Letter, we propose a fading model for which the Nakagami-*m* envelope-phase joint distribution is derived in an *exact and simple* manner.

*Nakagami-*m *phase-envelope joint distribution:* Let R and Θ be random variates representing, respectively, the envelope and phase of the Nakagami-*m* signal. The corresponding joint pdf $p_{R,\Theta}(r,\ \theta)$ is given by

$$p_{R,\Theta}(r,\ \theta)=\frac{m^m|\sin(2\theta)|^{m-1}\,r^{2m-1}}{2^{m-1}\Omega^m\Gamma^2(m/2)}\exp\left(-\frac{mr^2}{\Omega}\right)\qquad(1)$$

The envelope pdf $p_R(r)$ is the well-known formula

$$p_R(r)=\frac{2m^m r^{2m-1}}{\Omega^m\Gamma(m)}\exp\left(-\frac{mr^2}{\Omega}\right)\qquad(2)$$

and the phase pdf $p_\Theta(\theta)$ is given by

$$p_\Theta(\theta)=\frac{\Gamma(m)|\sin(2\theta)|^{m-1}}{2^m\Gamma^2(m/2)}\qquad(3)$$

Clearly

$$p_{R,\Theta}(r,\ \theta)=p_R(r)\times p_\Theta(\theta)\qquad(4)$$

i.e. phase and envelope are independent random variates.

Derivation of joint distribution: Let $N=R\exp(j\Theta)$ be the Nakagami-*m* signal, in which R is the envelope and Θ is the phase. Then

$$R^2=X^2+Y^2\quad\text{and}\quad\Theta=\tan^{-1}(Y/X)\qquad(5)$$

where

$$X=R\cos\Theta\quad\text{and}\quad Y=R\sin\Theta\qquad(6)$$

are the in-phase and quadrature components of the signal. For m integer

$$Z^2=\sum_{i=1}^{m}G_i^2\qquad(7)$$

where $Z=X$ or $Z=Y$, as required, and G_i are zero-mean Gaussian variates with variance $\Omega/2m$. Now, the aim is to obtain the distribution of Z. As an intermediate step, we find the pdf $p_W(w)$ of $W=Z^2$, which is known to follow a *chi-square* one *with m degrees of freedom*, i.e.

$$p_W(w)=\frac{m^{m/2}w^{(m/2)-1}}{\Omega^{m/2}\Gamma(m/2)}\exp\left(-\frac{mw}{\Omega}\right)\qquad(8)$$

The variate Z can be written as

$$Z=\operatorname{sgn}(Z)\times|Z|\qquad(9)$$

where $\operatorname{sgn}(\cdot)$ is the sign function. Note that $|Z|=\sqrt{W}$. Therefore

$$p_{|Z|}(|z|)=\frac{2m^{m/2}|z|^{m-1}}{\Omega^{m/2}\Gamma(m/2)}\exp\left(-\frac{mz^2}{\Omega}\right)\qquad(10)$$

Motivated by the fact that for $m=1$ the random variable Z must reduce to a Gaussian variate, we assume that positive and negative values of Z occur with identical probability. Therefore the pdf $pz(z)$ of Z is obtained from (10) as

$$p_Z(z)=\frac{m^{m/2}|z|^{m-1}}{\Omega^{m/2}\Gamma(m/2)}\exp\left(-\frac{mz^2}{\Omega}\right)\quad-\infty<z<\infty\qquad(11)$$

Using the transformation as in (5) then $p_{R,\Theta}(r,\theta)=|J|\,p_{X,Y}(x,y)$, where $|J|$ is the Jacobian of the transformation. Now $|J|=r$ and $p_{X,Y}(x,\ y)=p_X(x)\times p_Y(y)$, where the marginal pdf's are as in (11). By carrying out the required mathematical procedure (1) results. Although derived for integer values of m there are no mathematical constraints for these expressions to be used for any $m>0$.

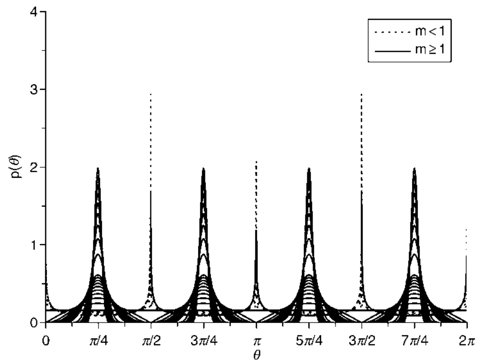

Fig. 1 *Nakagami-m phase distribution*

Discussion: We now compare the Nakagami-*m* phase distribution, as obtained here, with those of Hoyt and Rice. The Hoyt phase pdf is given by [5]

$$p_\Theta(\theta)=\frac{\sqrt{1-b^2}}{2\pi(1-b\cos(2\theta))}\qquad(12)$$

where $-1\le b\le1$ is the Hoyt fading parameter. The Rice phase pdf is obtained as

$$p_\Theta(\theta)=\frac{1+2\sqrt{\pi k}\,e^{4k\cos^2(\theta-\phi)}\cos(\theta-\phi)}{2\pi\,e^{4k}}\left(1+\operatorname{erf}(2\sqrt{k}\,\cos(\theta-\phi))\right)\qquad(13)$$

where $\operatorname{erf}(\cdot)$ is the error function, $k\ge0$ is the Rice fading parameter and ϕ is a phase which depends on the ratio of the quadrature dominant component and in-phase dominant component. If they are identical in modulus then $\phi=\pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$. Fig. 1 depicts the various shapes of Nakagami-*m* pdf $p_\Theta(\theta)$ against θ for $m=0.5,\ldots,\infty$. Note that for $m=0.5$ impulses tend to occur at $0,\ \pi/2,\ \pi$, and $3\pi/2$; for $m=1$ the distribution is uniform; and for $m\to\infty$ impulses tend to occur at

$\pm\pi/4$ and $\pm 3\pi/4$. For comparison, Fig. 2 shows the shapes of Hoyt and Rice phase distributions for the various ranges of their parameters. Note that both Figs. 1 and 2 bear similar shapes and are coincident at the limiting cases of their parameters. Fig. 3 shows the shapes of the Nakagami-m phase shapes in polar co-ordinates.

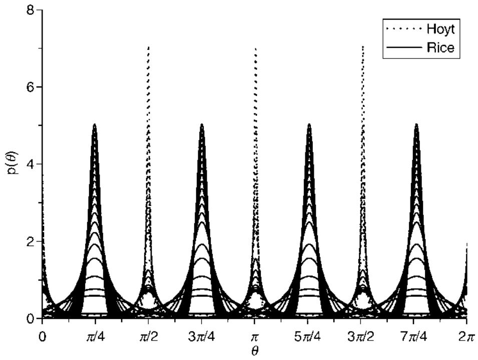

Fig. 2 *Rice and Hoyt phase distribution*

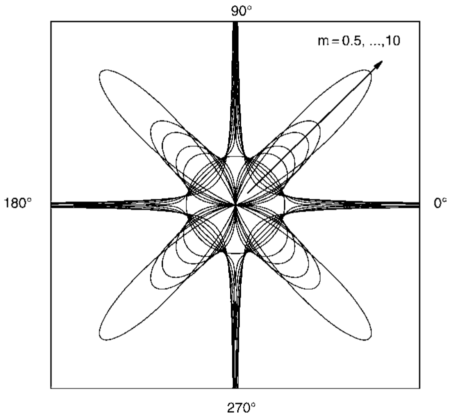

Fig. 3 *Nakagami-m phase distribution – polar co-ordinates*

Conclusions: In this Letter the Nakagami-m phase-envelope joint distribution has been obtained in an exact manner. Contrary to what is usually assumed, apart from the case $m=1$, the Nakagami-m phase distribution is not uniform. It bears shapes similar to those of Hoyt and Rice and coincides with them at limiting values of their parameters, for which their envelopes also coincide.

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